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1985 J. Phys. A: Math. Gen. 18 L261

(http://iopscience.iop.org/0305-4470/18/5/007)

J. Phys. A: Math. Gen. **18** (1985) L261-L266. Printed in Great Britain

LETTER TO THE EDITOR

Four-dimensional boson field theory†

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Received 15 October 1984

Abstract. We introduce the method of 'phantom fields' and show how it can be used to construct non-trivial, scalar, self-interacting, Euclidean boson fields in four-dimensional space. These fields satisfy all the required axioms except perhaps rotational invariance and are based on the continuum limit of lattice cut-off fields. The resulting field theory may not be asymptotically free.

A long standing problem has been the construction of a non-trivial, scalar, self-interacting boson field theory in four dimensions. In the context of the usual approaches, the existence of such a theory has been virtually excluded. Attention has centred on the $g_0 : \phi^4 :_4$ theory because higher polynomial powers in the interaction are not renormalisable (Itzykson and Zuber 1980). In fact, Newman (1979) has proven under mild assumptions that such a field cannot exist if its degree (which corresponds formally to the degree of the polynomial in the Hamiltonian) exceeds 4. More recently both numerical (Baker and Kincaid 1981) and theoretical (Aisenman 1981, 1982, Brydges *et al* 1982) evidence points strongly in the direction that the $g_0 : \phi^4 :_4$ theory itself is trivial, i.e., a generalised free field which has no scattering. As the $: \phi^2 :$ theory is of course just the free field, the usual approach is pretty well exhausted.

A way out of this dilemma has been discovered by Baker (1984b) in his study of limitations of critical index universality. We will call this discovery the 'method of phantom fields'. It bears a relation to the spirit of local effective Lagrangian theory (Symanzik 1982) and the theory of ultraviolet renormalons (Gross and Neveu 1974, Lautrup 1977, 't Hooft 1979). In the latter theory it is found (Parisi 1978, 1979, Bergère and David 1983) that they are proportional to the insertion of local irrelevant variables, as for example ϕ^6.

The models we wish to study are special cases of the continuous-spin Ising model. We work in Euclidean space, because if we are successful in satisfying, for example, Nelson's axioms (Nelson 1973), we are assured by his reconstruction theorem that a Minkowski space theory satisfying the, by now standard, Wightman axioms can be constructed from the Euclidean space one. We begin with a lattice cut-off version (hyper-simple-cubic lattice for convenience only) of the field theory. It is described by the partition function

$$Z = M^{-1} \int_{-\infty}^{+\infty} \cdots \int \prod_i \mathrm{d}\phi_i \exp\left[-\sum_i a^4 \left(\sum_{\{\delta\}} \frac{(\phi_i - \phi_{i+\delta})^2}{a^2} + m_0^2 \phi_i^2 + \lambda_0 : \mathscr{P}(\phi_i^2) : \right) \right], \qquad (1)$$

† Work performed under the auspices of the US DOE.

where M is a formal normalisation constant, i ranges over a portion of the integer lattice Z^4, the set $\{\delta\}$ is one half the set of nearest neighbours on the lattice, $\mathcal{P}(\phi^2)$ is a lower, semibounded, even monic polynomial of degree p and $:\phi^n:$ is the normal ordered product. As long as the lattice spacing a is greater than zero, the normal order product (Baker 1984a) is

$$:\phi^{2p}: = \sum_{j=0}^{p} \frac{(2p)!(-1)^j}{(2p-2j)!j!} 2^{-j} C^j \phi^{2p-2j}, \tag{2}$$

where C is the commutator $[\phi^-, \phi^+]$ and is proportional to a^{-2}. Using equation (2) we can re-express equation (1)

$$Z = M^{-1} \int_{-\infty}^{+\infty} \dots \int \prod_i \mathrm{d}\sigma_i \exp\left(K \sum_i \sum_{\{\delta\}} \sigma_i \sigma_{i+\delta} - \sum_i [\tilde{A}\sigma_i^2 + \tilde{\lambda}_0 P(\sigma_i^2)]\right), \tag{3}$$

where m is a different formal normalisation constant, P is again a polynomial of degree p, and

$$\sigma_i = a\phi_i(2/K)^{1/2}, \qquad \tilde{A} = K(4 + \tfrac{1}{2}a^2 m_0^2),$$

$$\tilde{\lambda}_0 = \lambda_0 (K/2)^p a^{4-2p}, \qquad P(x) = \sum_{j=1}^{p} a_j x^j, \qquad a_p = 1. \tag{4}$$

We impose the normalisation on the scale of σ_i

$$1 = \langle 1 \rangle = \langle \sigma_i^2 \rangle = \left(\int_{-\infty}^{+\infty} x^2 \exp[-\tilde{A}x^2 - \tilde{\lambda}_0 P(x^2)]\,\mathrm{d}x \right) \left(\int_{-\infty}^{+\infty} \exp[-\tilde{A}x^2 - \tilde{\lambda}_0 P(x^2)]\,\mathrm{d}x \right)^{-1}. \tag{5}$$

Now we wish to study the special case where, in terms of orders of magnitude of a, all the parameters in (3), i.e., K, $\tilde{A}$, $\tilde{\lambda}_0$, a_j, are of order unity. This prescription means, in terms of the field theory variables ϕ, that the coefficient of ϕ^{2n} is going to be proportional to $a^{2(n-2)}$. In other words the coefficient of ϕ^4 is of order unity, that of ϕ^6 of order a^2, ϕ^8 of order a^4, etc. This means that the coefficients of ϕ^6, $\phi^8, \dots$ nominally vanish in the continuum limit, $a \to 0$, and so we call them 'phantom fields'. Nevertheless, the normal ordered product of each of these terms contributes a coefficient of order unity to the coefficient of ϕ^4. For example, $a^2:\phi^6:= a^2\phi^6 - 15a^2C\phi^4 + \dots$. Since $C \propto a^{-2}$ we get a negative finite contribution to the coefficient of ϕ^4 in the continuum limit, which with a judicious choice of $\mathcal{P}$ could lead to a model in the nominal continuum limit with the opposite sign of the leading (ϕ^4) term from usual models. This model avoids Newman's proof (Newman 1979) of non-existence because, in this definition, the polynomial remains of nominal degree 4.

Briefly, we know rigorously (Baker 1984a) for models of this sort that, $a > 0$, for Dirichlet, periodic and free boundary conditions, the infinite volume (thermodynamic) limit exists for the free energy per unit volume and for the Schwinger functions (multipoint correlation functions). In addition, by using a general box size and arguments based on generalised Padé approximants (Baker 1984a), one can construct from the perturbation series in $\tilde{\lambda}_0$, the free energy and the Schwinger functions in the infinite volume limit ($a > 0$). By means of standard theorems on the convergence of a sequence of uniformly continuous functions, we can show that the infinite volume limit is continuous in the parameters of (2) for all positive real $\tilde{\lambda}_0$. In particular, the renormalised mass, m (second moment definition), is continuous and also the amplitude renormalisation constant. Further, as long as $a > 0$, the perturbation series in $\tilde{\lambda}_0$ in the infinite volume limit (Baker 1984a) is finite, term by term. The question of uniqueness of summability of this series is not yet rigorously resolved. By means of the FKG inequalities and superstable estimates Sokal (1981) has proven the cluster property

$$|\langle\sigma_A\sigma_B\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle|\leq C|\ln x|^{\gamma}x, \tag{6}$$

where $\sigma_A=\Pi_{a\in A}\,\sigma_a$, C and γ are appropriate constants and

$$x=\sum_{a\in A}\sum_{b\in B}(\langle\sigma_1\sigma_b\rangle-\langle\sigma_a\rangle\langle\sigma_b\rangle). \tag{7}$$

This inequality suffices to insure independence of the boundary conditions and to imply exponential decay of the multipoint correlations at large distances when that decay is true of the two-point correlations. Finally, from the existence of the appropriate transfer matrix, reflection positivity through lattice symmetry planes holds.

If we consider the limit $\tilde{\lambda}_0\to\infty$, then the single-site spin distribution function becomes a finite sum of Dirac delta functions. By an extension of the Peierls' argument (Griffiths 1972, Lebowitz and Gallavotti 1971) this model possesses a phase transition to a state with spontaneous magnetisation at sufficiently low temperature. Again general theorems on the limit of a converging sequence, through increasing box size, of continuous functions coupled with monotonicity in box size derived from Griffiths inequalities (Griffiths 1972) imply that an arbitrarily small limiting value of one over the susceptibility χ (using the untruncated definition) can be constructed in the thermodynamic limit by an appropriate adjustment of K as a function of lattice spacing and box size. Sokal's inequality (Sokal 1982) between the susceptibility and the correlations length, plus continuity, forces the same result for the correlation length, ξ (in units of the lattice spacing a). By known results for the free field model ($\tilde{\lambda}=0$), the same results hold for it. By continuity and monotonicity arguments, these conclusions suffice by usual methods (Baker 1984a) to prove mass and amplitude renormalisability for all positive real $\tilde{\lambda}_0$.

The above discussion is enough for us to gain control of the two-point function in the continuum limit. We now turn to the multipoint functions. For specificity, we will select $p=3$ (equation (4)) and so deal with a ϕ^6 theory. In this case we write out for convenience (3) as

$$Z=M^{-1}\int_{-\infty}^{+\infty}\cdots\int\prod_i \mathrm{d}\sigma_i\exp\left(K\sum_i\sum_{\{\delta\}}\sigma_i\sigma_{i+\delta}-\tilde{A}\sigma_i^2-\tilde{g}_0\sigma_i^4-\tilde{\lambda}_0\sigma_i^6\right). \tag{8}$$

From the first order in perturbation theory, with $\tilde{g}_0$, $\tilde{\lambda}_0$ taken as small and $\tilde{A}$ determined by equation (5), we obtain for the renormalised four-point coupling constant, expressed in terms of magnetic quantities (Baker 1984a),

$$g=\frac{-v\,\partial^2\chi/\partial H^2}{\chi^2\xi^4a^4}=K^{-2}[24\tilde{g}_0+360\tilde{\lambda}_0+30\tilde{\lambda}_0P_4(K)(24-672K$$
$$+7424K^2-36864K^3+69632K^4)]+O(\tilde{g}_0^2,\tilde{g}_0\tilde{\lambda}_0,\tilde{\lambda}_0^2), \tag{9}$$

where v is the specific hyper-volume per lattice site. For the hyper-simple-cubic lattice, $v=a^4$ and

$$P_4(K)=-1+(2\pi)^{-4}\int_0^{2\pi}\cdots\int\prod_{\tau=1}^4\mathrm{d}\theta_\tau\left[1-\left(2K\sum_{\tau=1}^4\cos\theta_\tau\right)^2\right]^{-1}. \tag{10}$$

The critical point, or continuum limit, is $K=\frac{1}{8}$, and $P_4(\frac{1}{8})$ is a finite positive constant.

If we choose

$$\tilde{g}_0 < -(15 + \tfrac{5}{4} P_4(\tfrac{1}{8}))\tilde{\lambda}_0, \tag{11}$$

then, to leading order in perturbation theory, $g < 0$ for all $0 < K \leqslant K_c = \frac{1}{8}$.

General power counting arguments are the same for this theory as for ϕ^4 theory (Itzykson and Zuber 1980). The extra internal momentum integrations, which lead to ultraviolet divergences in the usual way are exactly compensated by the factors of a^2 at each six-point vertex, and lead for $p = 3$, or larger, to a superficial degree of ultraviolet divergence of $4 - E$ where E is the number of external lines. This feature suggests that from the perturbation theoretic point of view no new primitive divergences occur for six or more external lines. Thus if we adjust our parameters to control the two- and four-point terms and insertions, by the usual theory (Itzykson and Zuber 1980) all will go well for the higher-order Schwinger functions.

It is worth pointing out that the contribution to g from the Feynman diagram of figure 1 is proportional to

$$(a^2\tilde{\lambda}_0)^2 \int_{-\pi/a}^{\pi/a} \cdots \int \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3\; \pi(\boldsymbol{k}_1)\pi(\boldsymbol{k}_2)\pi(\boldsymbol{k}_3)\pi(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3), \tag{12}$$

where

$$\pi(\boldsymbol{p}) = \left(m^2 + \frac{4}{a^2} \sum_{i=1}^{4} \sin^2(\tfrac{1}{2}\boldsymbol{p} \cdot \boldsymbol{e}_i \cdot a) \right)^{-1}. \tag{13}$$

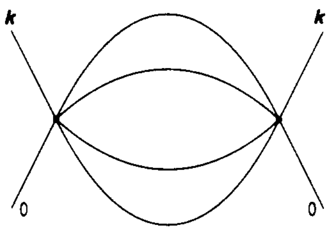

Figure 1. A phantom field Feynman diagram.

This term in the continuum limit is finite and independent of $\boldsymbol{k}$. Elaboration of this result suggests that the perturbation series will be rotationally invariant term-by-term so the continuum theory may also be but we have not proven it. Further, it would appear that the theory is not asymptotically free. The asymptotic freedom could be destroyed by the phantom fields, i.e., non-renormalisable, field self-interactions with strengths which vanish in the continuum limit, just as was the case for critical exponent universality (Baker 1984b) in three dimensions.

By the method of high-temperature series analysis, we have investigated, using the data of Baker and Kincaid (1981), the limit as $\tilde{\lambda}_0 \to \infty$ for $p = 3$ theories. In this class the limiting single-site spin distribution, equation (3), reduces to

$$\tfrac{1}{2}[\delta(s - S) + \delta(s + S)]S^{-2} + (1 - S^{-2})\delta(s), \qquad S \geqslant 1, \tag{14}$$

which has the moments $\langle s^{2n} \rangle = S^{2(n-1)}$. For $1 \leqslant S \leqslant \sqrt{2}$ the system obeys the Yang–Lee theorem (Lieb and Sokal 1981). For $S > \sqrt{3}$ and $K = 0$, the value of g as defined by equation (9) is negative instead of positive as it is for $1 \leqslant S < \sqrt{3}$. By increasing S

further, g remains negative for all $0 \leqslant K \leqslant K_c$ in analogy to equation (11) which holds for very small $\tilde{\lambda}_0$ and $\tilde{g}_0$.

Specifically, we have analysed the behaviour of g for $S = 2.1$ for the hyper-simple-cubic and hyper-body-centred-cubic lattices as well as a number of other cases. We have used the method of Padé analysis (Baker and Graves-Morris 1981), although such sophistication is not really necessary. We have first written g as a function of ξ^2 and then (Baker and Kincaid 1981) $\xi^2 = 0.1x/(1-x)$ so as to perform the analysis in x where the continuum limit, or critical point, corresponds to $x = 1$. We analyse the series for $x^2 g$, which are ($S = 2.1$)

$$\begin{aligned} -x^2 g(x) a^4/v &= 141 - 56.4x - 23.7876x^2 - 3.513\,72x^3 - 2.034\,7991x^4 \\ &\quad - 0.618\,310\,27x^5 - 0.756\,808\,43x^6 - 0.636\,873\,72x^7 \\ &\quad - 0.482\,039\,44x^8 - 0.337\,569\,92x^9 - 0.228\,675\,25x^{10} + \ldots \quad \text{HBCC} \\ &= 141 - 56.4x - 53.2152x^2 - 14.054\,88x^3 - 4.071\,843\,67x^4 \\ &\quad + 4.805\,3991x^5 + 2.367\,7591x^6 - 0.370\,990\,80x^7 \\ &\quad - 1.030\,7781x^8 - 0.397\,808\,93x^9 - 0.039\,295\,10x^{10} + \ldots \quad \text{HSC}. \end{aligned} \tag{15}$$

We estimate that $g(1) \simeq -25.9 \pm 0.1$ for HBCC, and $\simeq -18.7 \pm 0.3$ for the HSC lattice. The value of v is $0.5a^4$ for the HBCC and a^4 for the HSC.

We conclude, by these numerical results, (8), and continuity, that we can select $\tilde{g}_0(\tilde{\lambda}_0, a)$ in such a way that for each fixed $a > 0$ we can define a monotonically decreasing function $g(\tilde{\lambda}_0, a)$ which interpolates between $g(0, a) = 0$, and the numerical case given above. This function can be chosen in such a way that there exists a non-trivial, finite limit $g(\tilde{\lambda}_0, 0)$, $0 < \tilde{\lambda}_0 \leqslant \infty$. When combined with the previously mentioned power counting argument and our control of the two-point function, we believe that these procedures lead to the construction of a non-trivial, self-interacting, scalar, Euclidean, boson field theory in four dimensions.

We are pleased to acknowledge helpful discussions with Professor A S Wightman and J R Klauder.

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